3.4 – Rational Functions

Fractions of polynomials are called Rational Functions.

Eg: $f(x) = \frac{1}{x}$ is a rational function

Graphing gives

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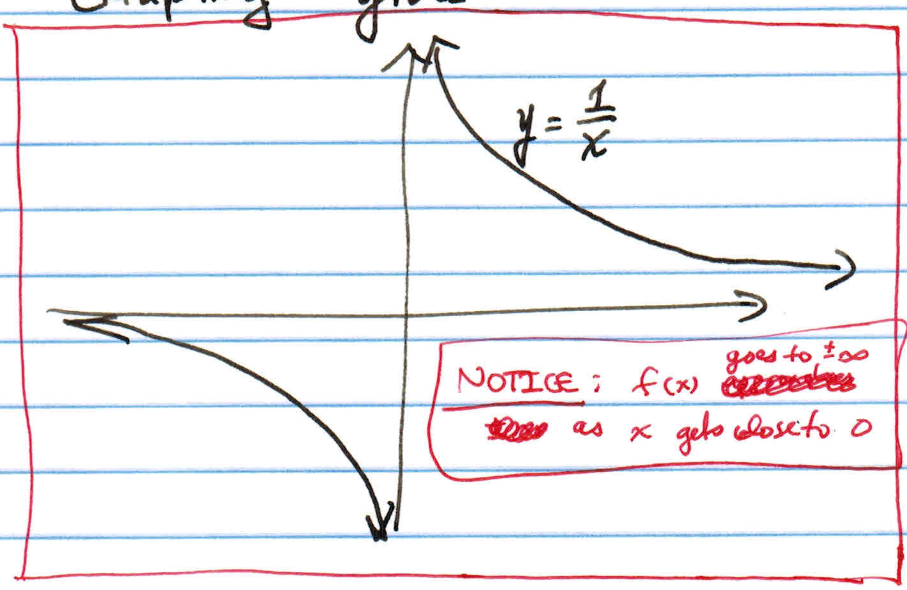
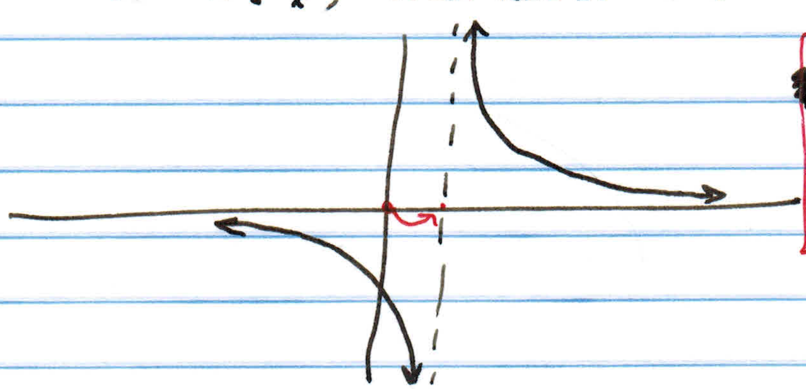


memorize!

Eg: Graph $f(x) = \frac{1}{x-1}$

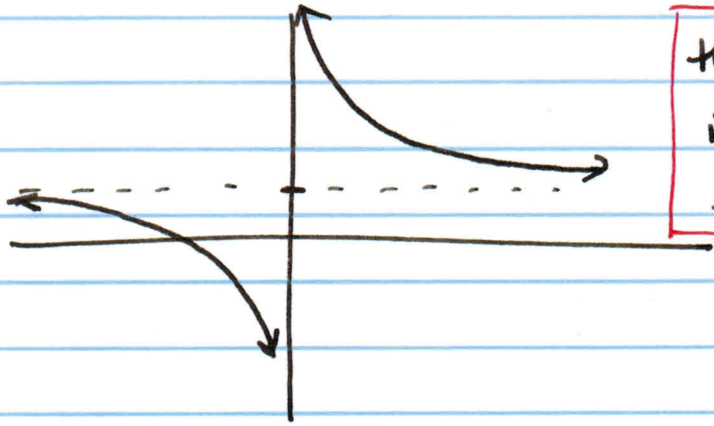
this is $y = \frac{1}{x}$, moved RIGHT one



the line $x = 1$ is called a vertical asymptote

Eg: Graph $f(x) = \frac{1}{x} + 1$

this is $y = \frac{1}{x}$, moved **UP** one



> the line $y = 1$ is called a **horizontal** asymptote

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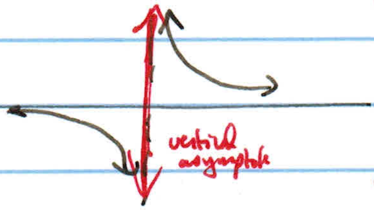
Define: the vertical line $x = a$ is a **vertical** asymptote, if

$f(x) \to \infty$ or $f(x) \to -\infty$

as x gets closer & closer to a



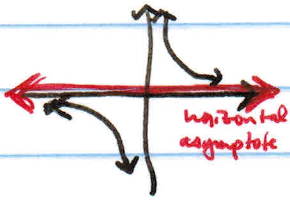
> **Eg:** $y = \frac{1}{x}$ has a **vertical** asymptote at $x = 0$

Define: the horizontal line $y = b$ is a horizontal asymptote IF

$f(x)$ goes to b

as $x \to \infty$ or $x \to -\infty$

Eg: $y = \frac{1}{x}$ has a horizontal asymptote at $y = 0$



Remember: when x is big positive/negative #

$$3x^2+x+2 \approx 3x^2$$

So: when x is big positive/negative #

$$\frac{3x^2+2}{4x^3+9} \approx \frac{3x^2}{4x^3} = \frac{3}{4x}$$

15 when x is big, $\frac{3}{4x} \approx 0$

$\Rightarrow \frac{3x^2+2}{4x^3+9}$ has a $\boxed{\text{horizontal asymptote [redacted] at } y=0}$

Eg: when x is big, [redacted]

$$f(x) = \frac{3x^3+2}{4x^3+9} \approx \frac{3x^3}{4x^3} = \frac{3}{4}$$

So:

$f(x)$ has a horizontal asymptote at $y = \frac{3}{4}$

Eg: when x is big

$$f(x) = \frac{3x^3+2}{4x^2+9} \approx \frac{3x^3}{4x^2} = \frac{3}{4}x$$

$\boxed{\text{goes to } \infty}$

$\Rightarrow$ when x is big, $f(x)$ looks like $\frac{3}{4}x$

To Graph a Rational Function

1. find the end behavior
   (this tells you if there are **horizontal** asymptotes)

2. find the local behavior

   10 A. find the zero's
      (the x-intercepts)
      Plot on # line as a dot

   B. find where it's undefined
      (these will be **vertical** asymptotes)
      Plot on # line with vertical dashed line

   Plot these on a # line!

3. find where f is positive/negative

   (use this to fill in the details)

Eg: Sketch the graph of

$$f(x) = \frac{3x}{x+2}$$

① End behavior

x is big $\Rightarrow$ $f(x) \approx \frac{3x}{x} = 3$

$\Rightarrow$ f has a horizontal asymptote at $y=3$

② local behavior

Ⓐ f(x) is zero

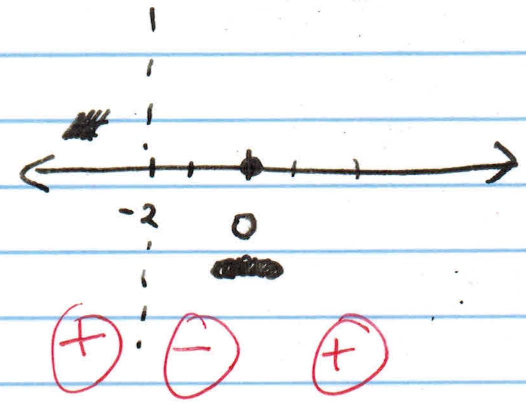
$$\frac{3x}{x+2} = 0$$

$$\Leftrightarrow$$

$$3x = 0$$

$$\Leftrightarrow$$

$$x = 0$$



Ⓑ f(x) is undefined

$\frac{3x}{x+2}$ undefined

$$\Leftrightarrow$$

$$x+2 = 0$$

$$\Leftrightarrow$$

$$x = -2$$

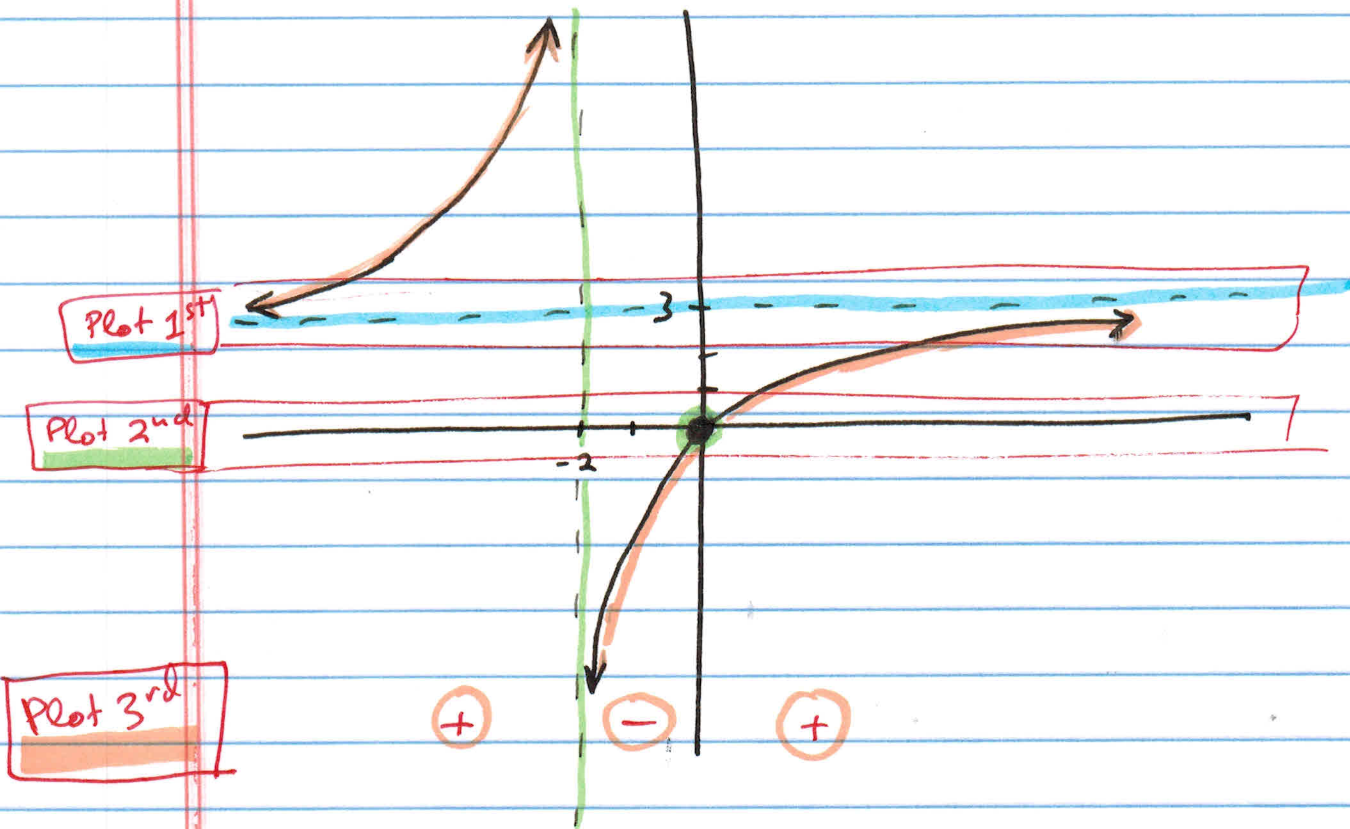
③ 3 intervals to check:

$x = -3 \Rightarrow f(-3) = \frac{3(-3)}{(-3)+2} = \frac{-9}{-1} = 9$

$x = -1 \Rightarrow f(-1) = \frac{3(-1)}{(-1)+2} = \frac{-3}{1} = -3$

$x = 1 \Rightarrow f(1) = \frac{3\cdot(1)}{(1)+2} = \frac{3}{3} = 1$

Putting it together



Eg: Sketch the graph of

$$f(x) = \frac{x^2-1}{x^2-9} = \frac{(x+1)(x-1)}{(x+3)(x-3)}$$

Step 1: End behavior

$$x \text{ is big} \Rightarrow f(x) \approx \frac{x^2}{x^2} = 1$$

$\Rightarrow$ f has a horizontal asymptote at $y = 1$

Step 2: Local behavior

(A) $f(x)$ hits the y-axis

$$\Leftrightarrow$$

$$\frac{x^2-1}{x^2-9} = \frac{(x+1)(x-1)}{(x+3)(x-3)} = 0$$

$$\Leftrightarrow$$

$x+1=0$ OR $x-1=0$

$$\Leftrightarrow$$

$x=-1$ or $x=1$

$\Rightarrow$ f hits the x-axis at $x=-1$ and $x=1$

(B) $f(x)$ is undefined

$$\Leftrightarrow$$

$\frac{(x+1)(x-1)}{(x+3)(x-3)}$ is undefined

$$\Leftrightarrow$$

$(x+3)(x-3)=0$

$\Leftrightarrow$

$x+3=0$ or $x-3=0$

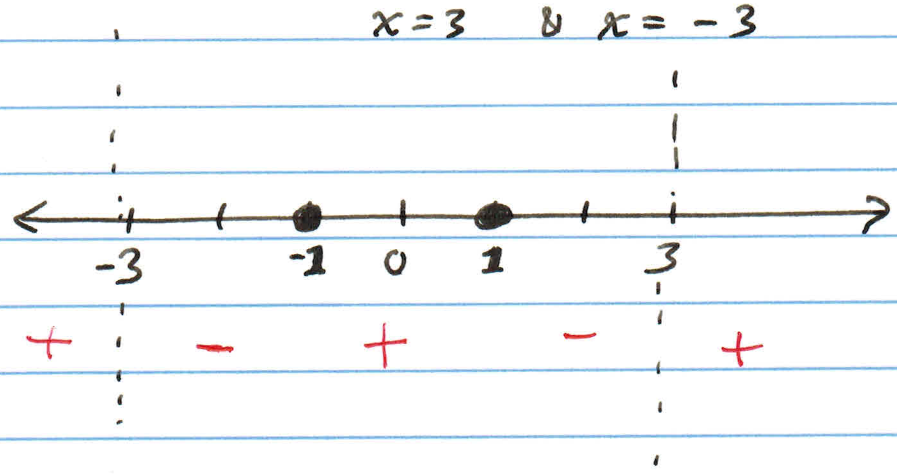
so f is undefined

$$\Leftrightarrow$$

$x=-3$ or $x=3$

$\Rightarrow$ $f(x)$ has <u>vertical</u> asymptotes at

$x = 3$ & $x = -3$



<u>step 3:</u> check when f is positive/negative

there are 5 intervals to check

$$f(-4) = \frac{(-4+1)(-4-1)}{(-4+3)(-4-3)} = \frac{(-3)(-5)}{(-1)(-7)} > 0$$

$$f(-2) = \frac{(-2+1)(-2-1)}{(-2+3)(-2-3)} = \frac{(-1)(-3)}{(1)(-5)} < 0$$

$$f(0) = \frac{(0+1)(0-1)}{(0+3)(0-3)} = \frac{(1)(-1)}{(3)(-3)} > 0$$

$$f(2) = \frac{(2+1)(2-1)}{(2+3)(2-3)} = \frac{3 \cdot 1}{5 \cdot (-1)} < 0$$

$$f(4) = \frac{(4+1)(4-1)}{(4+3)(4-3)} = \frac{5 \cdot 3}{7 \cdot 1} > 0$$

Graphing Gives